

# Inflationary $\alpha$ -attractors from $F(R)$ Gravity

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In this paper we study some classes of  $\alpha$ -attractors models in the Jordan frame and we find the corresponding  $F(R)$  gravity theory. We study analytically the problem at leading order and we investigate whether the attractor picture persists in the  $F(R)$  gravity equivalent theory. As we show, if the slow-roll conditions are assumed in the Jordan frame, the spectral index of primordial curvature perturbations and the scalar-to-tensor ratio are identical to the corresponding observational indices of the  $R^2$  model, a result which indicates that the attractor property is also found in the corresponding  $F(R)$  gravity theories of the  $\alpha$ -attractors models. Moreover, implicit and approximate forms of the  $F(R)$  gravity inflationary attractors are found.

PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k, 98.80.Cq, 11.25.-w

## I. INTRODUCTION

Inflationary cosmology is one of the two existing descriptions of the early Universe, in the context of which the theoretical inconsistencies of the Big Bang description of our Universe were successfully addressed [1–4], with the other alternative being bouncing cosmology [5–10]. The latest observational data coming from Planck [11] posed stringent constraints on inflationary models and verified the validity of some models, while rendering other models non-viable.

Recently, an interesting class of models was discovered in [12], called the  $\alpha$ -attractors models, with the characteristic property of these models being that the predicted spectral index of primordial curvature perturbations and the scalar-to-tensor ratio was identical for all the models, in the large  $N$  limit, where  $N$  is the  $e$ -foldings number. These models were later on studied in [13–21], and also for an earlier study of some  $\alpha$ -attractor-like potentials, see Ref. [22]. Well known inflationary models are special limiting cases of some  $\alpha$  attractors models, like for example the Starobinsky model [23, 24], the Higgs inflationary model [25]. To our opinion the most appealing property of the  $\alpha$ -attractors models is that these models have a large plateau in their potential, for large scalar field values, and in the small  $\alpha$  limit, and the potential of these models is asymptotically quite similar to the hybrid inflation scenario [26]. The hybrid inflation model introduced flat potentials in the research field of inflationary cosmology, and a crucial assumption was used, that the initial state of the scalar field did not correspond to the extremum of the scalar potential. Effectively, in the large field limit, all the  $\alpha$ -attractors models tend to some variant form of the hybrid inflation scenario, hence the very own idea of hybrid inflation is successful, in view of the observational data. The  $\alpha$ -attractors models originate from supergravity models, which also involve a controllable supersymmetry breaking at the minimum of the scalar potential [27]. In addition, in some cases, late-time acceleration can be accommodated in the context of  $\alpha$ -attractors models [28, 29].

In this paper we aim to investigate whether the attractor property also occurs in the Jordan frame, in terms of the corresponding  $F(R)$  gravity. In principle, since the curvature perturbation in the Jordan frame is invariant under a conformal transformation, and also since the tensor perturbations are invariant, someone would expect that the both the spectral index and also the scalar-to-tensor ratio in the Jordan frame, should be identical to the ones calculated for the Einstein frame. However, in the large field limit, the conformal transformation diverges, so in principle, the era where the scalar field reaches the plateau, makes the conformal transformation unbounded. This large field era corresponds to a pole in the scalar-field Jordan frame, and it is exactly the era where the  $\alpha$ -attractors models yield equivalent observational data. Motivated by this, the focus in this paper is to check explicitly whether the attractors property is shared by the  $F(R)$  gravity equivalent of these theories. We shall use the large field limit of the  $\alpha$ -attractors models and using well-known methods we shall find the Jordan frame  $F(R)$  gravity theory corresponding to the potential of the  $\alpha$ -attractors models. As we explicitly demonstrate, the general problem is not so trivial as it seems, since the analytic treatment is in general impossible to be performed, for general values of  $\alpha$ . In effect, we investigate certain convenient examples and also for various limits of the parameter  $\alpha$ , and as we show,

the attractors property holds true for the  $F(R)$  gravity equivalent theories, and more importantly, the models yield identical observational data to the  $R^2$  inflation model. This finds its explanation to the fact that the cosmological evolution during the slow-roll era is a quasi-de Sitter evolution.

But why there is a need to study the physics in different frames? This is a deep question, so now we shall try to answer this question, since this is our main motivation for the subject of this paper. In general, for every theoretical proposal in modified gravity, it is compelling to compare the results with the observational data. In this research line, the  $F(R)$  gravity Jordan frame and/or the Einstein frame, may provide a viable description of the observable Universe, however it is not for sure that a viable theory in the Jordan frame may give also a viable theory in the Einstein frame. In addition, the viability of a theoretical description does not come in hand with the physically convenient description. So the question is which of the two frames is the more physical one (at least, in some sense), or which of the two frames describes in a more appealing way the cosmic history of our Universe. To a great extent, the answer to this question depends on the compatibility of the resulting theory with the observational data. In addition, in principle there are quantities that should be the same in the Jordan and Einstein frames, and these are actually the quantities that are invariant under conformal transformations. For a quasi-de Sitter evolution, it is expected that the spectral index and the scalar-to-tensor ratio should be equivalent in the two frames, as it was shown in [30, 31]. However, this should be explicitly checked, since when neutron stars are studied, different results occur in the two frames [32]. In addition, a finite-time singularity of a certain type in one frame, does not correspond to the same type of singularity in the other frame [33, 34], since the conformal transformation becomes ill defined on the singular point. Also the presence of matter can lead to escalated complications between frames, since it is minimally coupled in the Jordan frame but it is non-minimally coupled in the corresponding Einstein frame. Also it may occur that the Universe is accelerating in one frame, but decelerates in the other [35]. Hence, these arguments essentially explain our motivation to study the attractor picture in the Jordan frame.

This paper is organized as follows: In section II we describe in brief the essential features of the  $\alpha$ -attractors models, and we demonstrate how the attractor property occurs. In section III, we address the same issue in the  $F(R)$  gravity equivalent theory and we demonstrate in detail how the attractor property occurs in this case, by studying analytically some characteristic examples and limiting cases. Finally, the conclusions follow in the end of the paper.

Also in this paper we will assume that the geometric background will be a flat Friedmann-Robertson-Walker metric, with line element,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2, \quad (1)$$

with  $a(t)$  denoting as usual the scale factor. Moreover, we assume that the connection is a symmetric, metric compatible and torsion-less affine connection, the so-called Levi-Civita connection. For the metric with line element that of Eq. (1), the Ricci scalar reads,

$$R = 6(2H^2 + \dot{H}), \quad (2)$$

with  $H$  denoting the Hubble rate  $H = \dot{a}/a$ . Also we use a units system such that  $\hbar = c = 8\pi G = \kappa^2 = 1$ .

## II. THE INFLATIONARY ATTRACTORS ESSENTIALS AND THE $F(R)$ GRAVITY DESCRIPTION

As we mentioned in the introduction, the terminology  $\alpha$ -attractor models refers to inflationary models with plateau potentials [12–20, 22]. These models include the  $R^2$  inflation model in the Einstein frame [23, 24], and the Higgs inflation model [25]. An essential feature in these models is the existence of a pole in the kinetic term of the non-canonical scalar field description. Usually the description using a non-canonical scalar field is called the Jordan frame description, so in order to avoid confusion with the  $F(R)$  description, we shall refer to the non-canonical scalar field Jordan frame as “ $\phi$ -Jordan frame” and to the  $F(R)$  Jordan frame simply as “Jordan frame”.

In the  $\phi$ -Jordan frame, the  $\alpha$ -attractors models have the following gravitational action [12],

$$\mathcal{S} = \sqrt{-g} \left( \frac{R}{2} - \frac{\partial_\mu \phi \partial^\mu \phi}{2(1 - \frac{\phi^2}{6\alpha})^2} - V(\phi) \right), \quad (3)$$

where  $R$  is the Ricci scalar and also we used units where the gravitational constant is  $G = 1$ . Notice that the action (3) contains a pole at  $\phi = \sqrt{6\alpha}$ , and this is of fundamental importance in the  $\alpha$ -attractor theories, since the order of the pole crucially affects the spectral index of primordial curvature perturbations  $n_s$ , while the residue of the pole affects the scalar-to-tensor ratio  $r$  [15]. By making the transformation,

$$\frac{d\phi}{1 - \frac{\phi^2}{6\alpha}} = d\varphi, \quad (4)$$

the non-canonical action of Eq. (3) is transformed into the canonical scalar field action,

$$\mathcal{S} = \sqrt{-g} \left( \frac{R}{2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\sqrt{6\alpha} \tanh(\frac{\varphi}{\sqrt{6\alpha}})) \right), \quad (5)$$

where the argument of the scalar potential easily follows by solving the transformation equation (4).

One of the most interesting features of the  $\alpha$ -attractors models is that at small  $\alpha$ , or equivalently at large  $\varphi$  values, the quite generic potentials  $V(\sqrt{6\alpha} \tanh(\frac{\varphi}{\sqrt{6\alpha}}))$  approach an infinitely long de Sitter plateau, which corresponds to the value of the non-canonical potential  $V(\phi)$  at the boundary  $V(\phi)|_{\pm\sqrt{6\alpha}}$ . The terminology attractors is justified due to the fact that regardless of the form of the potential, all the  $\alpha$ -attractor models lead to the same spectral index of primordial curvature perturbations  $n_s$  and to the same scalar-to-tensor ratio  $r$ , in the small  $\alpha$  limit, which have the following form,

$$n_s \simeq 1 - \frac{2}{N}, \quad r \simeq \frac{12\alpha}{N^2}, \quad (6)$$

where  $N$  is the  $e$ -foldings number. The purpose of this paper is to investigate if this attractor picture remains when one considers the  $F(R)$  gravity equivalent theory corresponding to the canonical scalar fields. The main reason behind the attractor picture in the Einstein frame is that the various generic potentials  $V(\sqrt{6\alpha} \tanh(\frac{\varphi}{\sqrt{6\alpha}}))$  have a similar limiting behavior in the small  $\alpha$  limit. We shall consider two classes of potentials, namely the T-models and the E-models, with the potential in the T-models case being of the following form,

$$V(\varphi) = \alpha\mu^2 \tanh^2(\frac{\varphi}{\sqrt{6\alpha}}), \quad (7)$$

where  $\mu$  is a positive number, freely chosen. In the case of the E-models, the potential has the following form,

$$V(\varphi) = \alpha\mu^2 \left( 1 - e^{-\sqrt{\frac{2}{3\alpha}}\varphi} \right)^{2n}, \quad (8)$$

with the parameter  $n$  being a positive number, not necessarily an integer. Note that for  $\alpha = n = 1$ , the potential (8) becomes,

$$V(\varphi) = \alpha\mu^2 \left( 1 - e^{-\sqrt{\frac{2}{3}}\varphi} \right)^2, \quad (9)$$

which is the Starobinsky model [23], so essentially the Starobinsky model is a subcase of the E-models. The scalar potential in the large  $\varphi$  limit becomes approximately equal to,

$$V(\varphi) \simeq \alpha\mu^2 \left( 1 - 4e^{-\sqrt{\frac{2}{3\alpha}}\varphi} \right), \quad (10)$$

while the E-model potential in the large  $\varphi$  limit becomes approximately equal to,

$$V(\varphi) \simeq \alpha\mu^2 \left( 1 - 2ne^{-\sqrt{\frac{2}{3\alpha}}\varphi} \right). \quad (11)$$

As it can be seen, the potentials of Eqs. (10) and (11) coincide when  $n = 2$ , but as we already mentioned the resulting spectral index  $n_s$  and the scalar-to-tensor ratio coincide for general  $n$ , and also the number  $n$  does not appear in the resulting expressions of  $n_s$  and  $r$ . Let us briefly demonstrate this issue, since it is of crucial importance when we compare the Einstein frame observational indices with the Jordan frame ones. As we will show, any difference should originate from the slow-roll conditions in the two frames. Let us consider the limiting case potential of Eq. (11), and in the following we shall focus on this potential, since almost all the cases we will study result to this potential in the small  $\alpha$  limit. The first two slow-roll indices  $\epsilon$  and  $\eta$  in the slow-roll approximation for a canonical scalar field are defined as follows,

$$\epsilon(\varphi) = \frac{1}{2} \left( \frac{V'(\varphi)}{V(\varphi)} \right)^2, \quad \eta(\varphi) = \frac{V''(\varphi)}{V(\varphi)}, \quad (12)$$

and also the  $e$ -foldings number  $N$  can also be expressed in terms of the potential when the slow-roll approximation is used, and it explicitly reads,

$$N \simeq \int_{\varphi}^{\varphi_i} \frac{V(\varphi)}{V'(\varphi)} d\varphi, \quad (13)$$

where  $\varphi_i$  is some initial value of the canonical scalar field. For the potential (11), the  $e$ -foldings number  $N$  can be expressed in terms of the canonical scalar field, and in the small  $\alpha$  limit, the resulting expression is,

$$N \simeq \frac{3\alpha e^{\sqrt{\frac{2}{3\alpha}}\varphi}}{4n}, \quad (14)$$

so by calculating the slow-roll indices and substituting the  $e$ -foldings number from Eq. (13), the slow-roll indices take the following form,

$$\epsilon \simeq \frac{3\alpha}{4N^2}, \quad \eta \simeq -\frac{1}{N}. \quad (15)$$

The spectral index of primordial curvature perturbations  $n_s$  and the scalar-to-tensor ratio  $r$  calculated for a canonical scalar field, are equal to,

$$n_s \simeq 1 - 6\epsilon + 2\eta, \quad r \simeq 16\epsilon, \quad (16)$$

so by substituting the slow-roll indices from the expressions (15), the resulting observational indices are,

$$n_s \simeq 1 - \frac{2}{N} - \frac{9\alpha}{2N^2}, \quad r \simeq \frac{12\alpha}{N^2}. \quad (17)$$

At large  $N$ , the observational indices of Eq. (17) coincide with the results of Eq. (6), so at leading order only the spectral index is independent of  $\alpha$  and also both the observational indices do not depend on the parameter  $n$ . This is exactly the attractor picture for the general class of the potentials, which have limiting form (11). Below we quote the three crucial conditions that need to hold true in order the attractor picture in the Einstein frame occurs:

- The small  $\alpha$  limit of the potential should be taken.
- The large  $N$  limit should be taken.
- The slow-roll approximation should hold true.

As we will show shortly, when these conditions hold true in the Jordan frame, then the attractor picture occurs in the Jordan frame too, with the difference that the observational indices have no  $\alpha$  dependence.

Let us start our Jordan frame considerations by firstly finding the vacuum  $F(R)$  gravity [36–39] that can generate potentials as in Eq. (11). This limiting case covers both the E-models and T-models in the small  $\alpha$  limit. Before proceeding to the main focus of this article, we recall some essential features of the connection between the Einstein and Jordan frame equivalent theories [22, 36, 37, 40–42]. Consider the following  $F(R)$  gravity action,

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} F(R), \quad (18)$$

where  $\hat{g}_{\mu\nu}$  is the metric tensor in the Jordan frame. Introducing the auxiliary field  $A$  in the Jordan frame action (18), the latter can be written as follows,

$$\mathcal{S} = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} (F'(A)(R - A) + F(A)). \quad (19)$$

By varying the action of Eq. (19) with respect to the scalar field  $A$ , it yields the solution  $A = R$ , and this proves the mathematical equivalence of the actions (19) and (18).

A crucial step in finding the Einstein frame canonical scalar-tensor theory corresponding to the  $F(R)$  gravity (18) is to perform a canonical transformation. It is important to note that the canonical transformation should not contain the parameter  $\alpha$ , see the discussion in the Appendix on this issue. The canonical transformation that connects the Einstein and Jordan frames is the following,

$$\varphi = \sqrt{\frac{3}{2}} \ln(F'(A)) \quad (20)$$

with  $\varphi$  being the canonical scalar field in the Einstein frame. Upon conformally transforming the Jordan frame metric  $\hat{g}_{\mu\nu}$  as follows,

$$g_{\mu\nu} = e^{-\varphi} \hat{g}_{\mu\nu} \quad (21)$$

where  $g_{\mu\nu}$  denotes the Einstein frame metric, we obtain the following Einstein frame canonical scalar field action,

$$\begin{aligned}\tilde{S} &= \int d^4x \sqrt{-g} \left( R - \frac{1}{2} \left( \frac{F''(A)}{F'(A)} \right)^2 g^{\mu\nu} \partial_\mu A \partial_\nu A - \left( \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \right) \right) \\ &= \int d^4x \sqrt{-g} \left( R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right)\end{aligned}\quad (22)$$

The Einstein frame potential  $V(\varphi)$  of the canonical scalar field  $\varphi$  is equal to,

$$V(\varphi) = \frac{1}{2} \left( \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \right) = \frac{1}{2} \left( e^{-\sqrt{2/3}\varphi} R \left( e^{\sqrt{2/3}\varphi} \right) - e^{-2\sqrt{2/3}\varphi} F \left[ R \left( e^{\sqrt{2/3}\varphi} \right) \right] \right). \quad (23)$$

The Ricci scalar as a function of the scalar field can be found by solving Eq. (20) with respect to  $A$ , having in mind of course the equivalence of  $R$  and  $A$ . It is straightforward to obtain the  $F(R)$  gravity that generates a specific potential, by simply combining Eqs. (23) and (20). Indeed, by taking the derivative of both sides of Eq. (23), with respect to the Ricci scalar, and also due to the fact that  $\frac{d\varphi}{dR} = \sqrt{\frac{3}{2}} \frac{F''(R)}{F'(R)}$ , we obtain the following relation, which is crucial for the analysis that follows,

$$R F_R = 2 \sqrt{\frac{3}{2}} \frac{d}{d\varphi} \left( \frac{V(\varphi)}{e^{-2(\sqrt{2/3})\varphi}} \right) \quad (24)$$

with  $F_R = \frac{dF(R)}{dR}$ . The above differential equation (24) combined with the solution of Eq. (20) with respect to  $R$ , will provide us with the  $F(R)$  gravity that generates some of the  $\alpha$ -attractors potential we presented earlier. For illustrative purposes let us see how the  $F(R)$  reconstruction method works, given the Einstein frame. Consider the Starobinsky potential (9), so by substituting this in Eq. (24), and also using the fact that  $F_R = e^{\sqrt{\frac{2}{3}}\varphi}$ , we obtain the following algebraic equation,

$$F_R R - (4F_R^2 \mu^2 - 4F_R \mu^2) = 0, \quad (25)$$

which has the solution,

$$F_R = \frac{4\mu^2 + R}{4\mu^2}, \quad (26)$$

so by integrating with respect to  $R$  we obtain the well-known  $R^2$  model, which is  $F(R) = R + \frac{R^2}{8\mu^2}$ . Note that the latter result gives implicitly the corresponding  $F(R)$ -gravity alpha-attractor.

### III. $F(R)$ GRAVITY DESCRIPTION: SOME EXAMPLES FOR SPECIFIC AND LIMITING CASES OF $\alpha$

#### A. The Case $\alpha = 1/4$

By using the reconstruction method we presented we will investigate which  $F(R)$  gravities can generate the  $\alpha$ -attractors potential we discussed earlier. We shall be interested in the large  $\varphi$  values which correspond to the inflationary de Sitter plateau in the Einstein frame, or near the pole at  $\phi = \sqrt{6}$  in the  $\phi$ -Jordan frame. Suppose that  $\alpha$  is not specified, so by substituting the potential (11) in Eq. (24), we obtain the following algebraic equation,

$$F_R R - 4\alpha\mu^2 F_R^{-\left(\sqrt{\frac{1}{\alpha}}-2\right)} \left( F_R^{\sqrt{\frac{1}{\alpha}}} + \left( \sqrt{\frac{1}{\alpha}} - 2 \right) n \right) = 0. \quad (27)$$

However for general  $\alpha$  it is a rather formidable task to solve the algebraic equation (27), so we shall specify the value of  $\alpha$  for various interesting cases. An interesting case, and one of the few that can be analytically solved, is for  $\alpha = 1/4$  since the parameter  $\alpha$  is smaller than unity. Consider that  $\alpha = 1/4$ , in which case the algebraic equation (27) is simplified as follows,

$$F_R R - F_R^2 \mu^2 = 0, \quad (28)$$

and the non-trivial solution to (28) is  $F_R(R) = \frac{R}{\mu^2}$ , therefore, the resulting  $F(R)$  gravity is,

$$F(R) = \frac{R^2}{2\mu^2} + \Lambda. \quad (29)$$

The integration constant  $\Lambda$  can only be specified if we follow the inverse reconstruction procedure and we identify the resulting potential with (11), for  $\alpha = 1/4$ . Indeed, by using Eq. (20), we obtain that  $R = \mu^2 e^{\sqrt{\frac{2}{3}}\varphi}$ , so by combining this with the resulting  $F(R)$  gravity (29) and by substituting in the first equation in Eq. (23), we obtain the following potential,

$$V(\varphi) = \frac{\mu^2}{4} \left( 1 - \frac{2\Lambda}{\mu^2} e^{-2\sqrt{\frac{2}{3}}\varphi} \right). \quad (30)$$

The potential (30) has to be identical to the one in Eq. (11), so the parameter  $\Lambda$  is  $\Lambda = n\mu^2$ . Having the  $F(R)$  gravity equivalent theory of the  $\alpha$ -attractor potential (11), we can calculate the slow-roll indices and the corresponding observational indices in the Jordan frame and see whether the attractor picture remains, as in the Einstein frame.

Let us start by finding an approximate expression for the Hubble rate at early times, as a function of the cosmic time  $t$ . In order to do this we will need the cosmological equations for the FRW metric (1) in the case a general  $F(R)$  gravity is used. By varying the action (18), with respect to the corresponding metric, we obtain the following cosmological equations,

$$\begin{aligned} 6F_R H^2 &= F_R R - F - 6H \dot{F}_R, \\ -2\dot{H} F_R &= \ddot{F}_R - H \dot{F}_R, \end{aligned} \quad (31)$$

so by using these and the slow-roll approximation, we will be able to obtain an approximate form for the Hubble rate during the slow-rolling phase of inflation. We shall use the first equation in Eq. (31), so by substituting the expressions for  $F_R$  and  $F(R)$  from Eq. (29) and also the expression for the Ricci scalar (2), we obtain the following differential equation,

$$36H''(t) + \frac{\mu^4 n - 18H'(t)^2}{H(t)} + 108H(t)H'(t) = 0. \quad (32)$$

The only dominant term during the slow-roll phase is the last one, so by solving it we obtain  $H(t) = H_0$ , which describes a de Sitter solution. However, it can be checked that the exact de Sitter solution is not a solution to the following equation,

$$2F(R) - RF'(R) = 0, \quad (33)$$

when the  $F(R)$  gravity is equal to the one appearing in Eq. (29), so this means that the approximate solution  $H(t) \simeq H_0$  is a leading order result and more terms are needed in order to better describe the solution. The solution  $H(t) = H_0$  during the slow-roll era where  $F'(R) \gg 1$  can also be verified by using well-known results in the literature [36], where for an  $F(R)$  gravity of the form  $F(R) = C + \alpha R^n$ ,  $n > 0$ , the first slow-roll index during the slow-rolling phase is calculated to be,

$$\epsilon_1 = \frac{2-n}{(n-1)(2n-1)}, \quad (34)$$

which is identical to the slow-roll index corresponding to an  $F(R) = R + \alpha R^n$  gravity. Therefore for  $n = 2$  the first slow-roll index is zero which implies that  $\dot{H}(t) = 0$  and hence the Hubble rate  $H(t) \simeq H_0$  describes the evolution during the slow-roll phase. However, as in the case we described earlier, the exact de Sitter solution is not a solution to the equation (33) for both the  $C + \alpha R^n$  and the  $R + \alpha R^n$  model, even for  $n = 2$ . Therefore we seek a leading order quasi-de Sitter evolution, exactly as in the case of the  $R^2$  gravity model. So we differentiate the first equation in Eq. (31), with respect to the cosmic time, and we get the following differential equation,

$$\begin{aligned} 6H'(t)R'(t)F''(R(t)) + 6H(t)^2R'(t)F''(R(t)) - R(t)R'(t)F''(R(t)) + 6H(t)(R'(t)^2F'''(R(t)) + R''(t)F''(R(t))) \\ + 12H(t)F'(R(t))H'(t) + F'(R(t))R'(t) - F'(R(t))R'(t) = 0. \end{aligned} \quad (35)$$

In effect, by substituting the  $F(R)$  gravity and its higher derivatives with respect to the Ricci scalar, we obtain the following differential equation,

$$\frac{36H'''(t)}{H(t)} + 108H''(t) + \frac{216H'(t)^2}{H(t)} = 0. \quad (36)$$

During the slow-roll phase, the first and last terms are subdominant, since  $H(t) \gg H^{(3)}(t)$  and  $H'(t) \ll H(t)$ , plus the last term contains a higher power of  $H'(t)$ . So the only term that yields the leading order solution is the second term, so by solving the resulting differential equation we obtain the Hubble rate during the slow-roll phase which is,

$$H(t) = H_0 - H_i(t - t_k), \quad (37)$$

which is a quasi de Sitter evolution, and  $H_0, H_i$  are arbitrary integration constants. Note that  $t_k$  is chosen to be the time that the horizon crossing occurred, at which time the comoving wavenumber satisfied  $k = a(t)H(t)$ , with  $a(t)$  being the scale factor. Also the minus sign in the Hubble evolution (37) has been chosen in order the first slow-roll parameter at the end of inflation has a positive sign. Having the approximate expression for the Hubble rate during the slow-rolling phase, will enable us to calculate the observational indices in the  $F(R)$  gravity case. The general expressions of the slow-roll indices for an  $F(R, \phi)$  gravity with gravitational action (setting  $\kappa = 1$ ),

$$\mathcal{S} = \int d^4x \sqrt{-g} \left( F(R, \phi) - \frac{\omega(\phi)}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right), \quad (38)$$

are equal to [43],

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\dot{F}'(R, \phi)}{2HF'(R, \phi)}, \quad \epsilon_4 \simeq \frac{\dot{E}}{2HE}, \quad (39)$$

where  $E$  is equal to,

$$E = F'(R, \phi)\omega(\phi) + \frac{3\dot{F}'(R, \phi)^2}{2\dot{\phi}^2}. \quad (40)$$

Specifying now, the slow-roll indices as functions of the Hubble rate for a general  $F(R)$  gravity, are equal to [43],

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = 0, \quad \epsilon_3 \simeq \epsilon_1, \quad \epsilon_4 \simeq -3\epsilon_1 + \frac{\dot{\epsilon}_1}{H(t)\epsilon_1}, \quad (41)$$

and the spectral index of primordial curvature perturbations  $n_s$  and the scalar-to-tensor ratio  $r$  reads,

$$n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4 \simeq 1 - \frac{2\dot{\epsilon}_1}{H(t)\epsilon_1}, \quad r = 48\epsilon_1^2. \quad (42)$$

The slow-roll approximation breaks down at first order when the first slow-roll parameter becomes of order one, that is  $\epsilon_1 \simeq \mathcal{O}(1)$ , so at that point, say at  $t = t_f$ , assume that the Hubble rate is  $H(t_f) = H_f$ . From the expression of the first slow-roll parameter  $\epsilon_1$ , we get  $1 \simeq \frac{H_i}{H_f^2}$ , so  $H_f \simeq \sqrt{H_i}$ . Then from Eq. (37) we obtain that,

$$H_f - H_0 \simeq -H_i(t_f - t_k), \quad (43)$$

so by substituting  $H_f$  we get,

$$t_f - t_k = \frac{H_0}{H_i} - \frac{\sqrt{H_i}}{H_i}. \quad (44)$$

Since  $H_0$  and  $H_i$  are expected to be of the same order during the slow-roll and also it is expected that these parameters have quite large values. In effect, the second term in Eq. (44) is subdominant, and therefore we have,

$$t_f - t_k \simeq \frac{H_0}{H_i}. \quad (45)$$

In order to introduce the  $e$ -foldings number into the calculation, we use the relation that expresses the  $e$ -folding number as a function of the Hubble rate,

$$N = \int_{t_k}^{t_f} H(t) dt, \quad (46)$$

calculated from the horizon crossing time until the end of inflation time. Substituting Eq. (37) in Eq. (46) we get,

$$N = H_0(t_f - t_k) - \frac{H_i(t_f - t_k)^2}{2}, \quad (47)$$

so by using (45) we finally obtain,

$$N = \frac{H_0^2}{2H_i}. \quad (48)$$

In effect, we have at leading order,

$$t_f - t_k \simeq \frac{2N}{H_0}, \quad (49)$$

so by calculating the slow-roll indices, we easily find that the spectral index and the scalar-to-tensor ratio are equal to,

$$n_s \simeq 1 - \frac{4H_i}{\left(H_0 - \frac{2H_i N}{H_0}\right)^2}, \quad r = \frac{48H_i^2}{\left(H_0 - \frac{2H_i N}{H_0}\right)^4}. \quad (50)$$

In the large  $N$  limit the observational indices read,

$$n_s \simeq 1 - \frac{H_0^2}{H_i N^2}, \quad r = \frac{3H_0^4}{H_i^2 N^4}, \quad (51)$$

so by substituting Eq. (48) in Eq. (51) we get,

$$n_s \simeq 1 - \frac{2}{N}, \quad r \simeq \frac{12}{N^2}. \quad (52)$$

This result is identical to the one of  $R^2$ -inflation [23] or Higgs inflation [25] due to the well-established equivalence of spectral index and of the scalar-to-tensor ratio in the Einstein and  $F(R)$  frames [31], when the slow-roll approximation is assumed. Therefore, we demonstrated that even for a general value of the parameter “ $\alpha$ ”, in the Jordan frame, the general  $\alpha$ - $F(R)$  gravity models result to the same spectral index and scalar-to-tensor ratio, therefore the attractor picture remains in the Jordan frame too, at least at leading order. We need to note that a crucial assumption for the calculation was that a slow-roll era is realized in the model and also the large  $N$  limit was taken in the end. Finally, also note that graceful exit in this theory is achieved in the same way as in  $R^2$  inflation, or may be generated by growing curvature perturbations.

The calculations we performed here could be performed because the  $\alpha = 1/4$  case was easy to deal analytically. However for general values of  $\alpha$  this is not possible. Take for example the  $\alpha = 1/9$  case, in which case, the algebraic equation (27) becomes,

$$RF_R - \frac{4\mu^2 (F_R^3 + n)}{9F_R} = 0, \quad (53)$$

and the solution to this equation is,

$$F_R = \frac{3R}{4\mu^2} - \frac{9R^2}{4\mu^2 \sqrt[3]{8\sqrt{16\mu^{12}n^2 - 27\mu^6 n R^3} + 32\mu^6 n - 27R^3}} - \frac{\sqrt[3]{8\sqrt{16\mu^{12}n^2 - 27\mu^6 n R^3} + 32\mu^6 n - 27R^3}}{4\mu^2}. \quad (54)$$

As it is obvious, this equation cannot be solved analytically with respect to  $R$  and hence, we cannot find an analytic expression for the potential.

Before closing we need to examine whether the value for  $\alpha$  we chose, namely  $\alpha = 1/4$ , yields viable results in the Einstein frame. So we examine the Einstein frame observational indices of Eq. (17), for  $\alpha = 1/4$ . For the set of values  $(N, \alpha) = (60, 1/4)$ , we obtain  $n_s \simeq 0.966667$  and also  $r \simeq 0.000833333$ , which are compatible with the Planck data which constraint  $n_s$  and  $r$  as follows [11],

$$n_s = 0.9644 \pm 0.0049, \quad r < 0.10. \quad (55)$$

Also for the set  $(N, \alpha) = (50, 1/4)$  we get,  $n_s \simeq 0.966667$  and also  $r \simeq 0.0012$ , which are also in agreement with the Planck data of Eq. (55). Hence, for all the physically relevant values of the  $e$ -foldings number  $N$ , which lie in the interval  $N = (50, 60)$ , the value  $\alpha = 1/4$  makes the Einstein frame observables compatible with the Planck data.



### B. The Case $\alpha = 4$

Another case that can be treated analytically is the case  $\alpha = 4$ , in which case the algebraic equation (27) becomes,

$$F_R R - 16F_R^{3/2} \mu^2 \left( \sqrt{F_R} - \frac{3n}{2} \right) = 0, \quad (56)$$

and there are two non-trivial solutions to Eq. (56), which are,

$$F_R = \frac{18\mu^4 n^2 + 6\sqrt{9\mu^8 n^4 + \mu^6 n^2 R} + \mu^2 R}{16\mu^4}, \quad (57)$$

$$F_R = \frac{18\mu^4 n^2 - 6\sqrt{9\mu^8 n^4 + \mu^6 n^2 R} + \mu^2 R}{16\mu^4}, \quad (58)$$

but the only solution which can yield the potential (11) is that of Eq. (57), as we now show. Indeed, the  $F(R)$  gravity corresponding to Eq. (57) is equal to,

$$F(R) = \frac{9n^2 \sqrt{\mu^6 n^2 (9\mu^2 n^2 + R)}}{4\mu^2} + \frac{R \sqrt{\mu^6 n^2 (9\mu^2 n^2 + R)}}{4\mu^4} + \frac{9n^2 R}{8} + \frac{R^2}{32\mu^2} + \Lambda, \quad (59)$$

where  $\Lambda$  is a constant the value of which will be determined shortly. Correspondingly, the  $F(R)$  gravity corresponding to Eq. (58) is equal to,

$$F(R) = -\frac{9n^2 \sqrt{\mu^6 n^2 (9\mu^2 n^2 + R)}}{4\mu^2} - \frac{R \sqrt{\mu^6 n^2 (9\mu^2 n^2 + R)}}{4\mu^4} + \frac{9n^2 R}{8} + \frac{R^2}{32\mu^2} + \Lambda. \quad (60)$$

The Einstein frame potential corresponding to the  $F(R)$  gravities above can be easily calculated by using the canonical transformation relation (20), which for both the  $F(R)$  gravities of Eqs. (59) and (60) yields the following two solutions,

$$R = 8\mu^2 e^{\frac{\varphi}{\sqrt{6}}} \left( 3n + 2e^{\frac{\varphi}{\sqrt{6}}} \right), \quad (61)$$

$$R = 8\mu^2 e^{\frac{\varphi}{\sqrt{6}}} \left( 3n - 2e^{\frac{\varphi}{\sqrt{6}}} \right), \quad (62)$$

Consider first the case for which the  $F(R)$  gravity is given by (60), so by combining Eqs. (61) and (23), the resulting Einstein frame potential is,

$$V(\varphi) = -\frac{1}{2}\Lambda e^{-2\sqrt{\frac{2}{3}}\varphi} + 4\mu^2 - \frac{1}{8}27\mu^2 n^4 e^{-2\sqrt{\frac{2}{3}}\varphi} - 27\mu^2 n^3 e^{-\sqrt{\frac{2}{3}}\varphi} - 36\mu^2 n^2 e^{-\sqrt{\frac{2}{3}}\varphi} - \mu^2 n e^{-\frac{\varphi}{\sqrt{6}}}, \quad (63)$$

which cannot be equal to the potential of Eq. (11), regardless of the value of the parameter  $\Lambda$ . The same applies if we choose the solution (62), for the  $F(R)$  gravity of Eq. (60). So the only  $F(R)$  gravity with physical interest which reproduces correctly the Einstein frame potential (23) is that of Eq. (59), which for the solution (62) it becomes,

$$V(\varphi) = -\frac{1}{2}\Lambda e^{-2\sqrt{\frac{2}{3}}\varphi} + 4\mu^2 + \frac{27}{8}\mu^2 n^4 e^{-2\sqrt{\frac{2}{3}}\varphi} - 8\mu^2 n e^{-\frac{\varphi}{\sqrt{6}}}, \quad (64)$$

so by choosing the constant  $\Lambda$  to be equal to  $\Lambda = \frac{27\mu^2 n^4}{4}$ , the potential (64) is identical to the potential of Eq. (23). Now let us find an approximate expression for the Hubble rate during the slow-roll era, and in order to do this, we use the expressions for the  $F(R)$  gravity and its derivative given in Eqs. (59) and (57), and plug this in the first equation of Eq. (31), so after some algebra we obtain,

$$\begin{aligned} & \frac{\Lambda}{H(t)^2} + \frac{27H'(t)}{4\mu^2} + \frac{3\sqrt{3}H(t)\sqrt{\mu^6 n^2}}{2\mu^4} + \frac{9\sqrt{3}n^2\sqrt{\mu^6 n^2}}{2\mu^2 H(t)} + \frac{27n^2}{4} \\ & + \frac{9\sqrt{3}\sqrt{\mu^6 n^2}H''(t)}{8\mu^4 H(t)^2} + \frac{9H''(t)}{4\mu^2 H(t)} + \frac{3\sqrt{3}\sqrt{\mu^6 n^2}H'(t)}{\mu^4 H(t)} + \frac{9H'(t)^2}{8\mu^2 H(t)^2} = 0, \end{aligned} \quad (65)$$

and therefore in the slow-roll limit, this differential equation becomes,

$$\frac{27H'(t)}{4\mu^2} + \frac{3\sqrt{3}H(t)\sqrt{\mu^6 n^2}}{2\mu^4} + \frac{27n^2}{4} = 0. \quad (66)$$

The above differential equation can be solved analytically, so the resulting solution is,

$$H(t) \simeq \mathcal{C}e^{-H_i(t-t_k)} - H_0, \quad (67)$$

where the parameter  $\mathcal{C}$  is an arbitrary integration constant,  $t_k$  is the horizon crossing time, while  $H_i$  and  $H_0$  are equal to,

$$H_0 = \frac{3}{2}\sqrt{3}\mu n, \quad H_i = \frac{2\mu n}{3\sqrt{3}}. \quad (68)$$

Note that since the cosmic time in the solution (67) takes values of the order  $\sim \mathcal{O}(10^{-20})\text{sec}$ , the exponential is of the order  $\sim \mathcal{O}(1)$ , so practically the evolution is a nearly de Sitter evolution. Also, in order for the Hubble rate to have negative values, the parameter  $\mathcal{C}$  must satisfy  $\mathcal{C} \gg H_0$ . However, the evolution is actually a quasi de Sitter expansion, and this can be seen by expanding the exponential in (67) in powers of the cosmic time, so the evolution becomes,

$$H(t) \simeq \mathcal{C} - H_0 - \mathcal{C}H_i(t - t_k). \quad (69)$$

Hence by having the evolution (69) at hand, we can easily calculate the observational indices. Following the steps of the  $\alpha = 1/4$  case, the spectral index of the primordial curvature perturbations in terms of the  $e$ -foldings number  $N$ , is at leading order,

$$n_s \simeq 1 - \frac{4CH_i}{\left(C \left(\frac{H_i N}{\mathcal{C} - H_0} - 1\right) + H_0\right)^2}, \quad (70)$$

while the scalar-to-tensor ratio is,

$$r \simeq \frac{48\mathcal{C}^2 H_i^2}{\left(-\frac{\mathcal{C}H_i N}{\mathcal{C} - H_0} + \mathcal{C} - H_0\right)^4}. \quad (71)$$

Therefore, in the large  $N$  limit, the observational indices read,

$$n_s \simeq 1 - \frac{2}{N}, \quad r = \frac{12}{N^2}, \quad (72)$$

where we have used the fact that during the slow-roll era,  $N \simeq \frac{(\mathcal{C}-H_0)^2}{2\mathcal{C}H_i}$ . So the resulting observational indices are identical to the ones of Eq. (52), which means that the  $R^2$  model in the slow-roll approximation is the attractor of this  $\alpha$  model too, in the large  $N$  limit of course.

As we did in the previous case, we need to examine whether the value  $\alpha = 4$ , yields viable results also in the Einstein frame. So we examine the Einstein frame observational indices of Eq. (17), for  $\alpha = 4$  in this case. For the set of values  $(N, \alpha) = (60, 4)$ , we obtain  $n_s \simeq 0.966667$  and also  $r \simeq 0.0133333$ , which are compatible with the Planck data of Eq. (55). In addition, for the set of values  $(N, \alpha) = (50, 1/4)$  we obtain  $n_s \simeq 0.966667$ , and the predicted scalar-to-tensor ratio is  $r \simeq 0.0192$ , so these are also compatible to the Planck constraints. Therefore, the case  $\alpha = 4$  yields physically viable observational indices, for all the physically relevant values of the  $e$ -foldings number  $N$ , which lie in the interval  $N = (50, 60)$ . However, the case  $\alpha > 4$  is somewhat more involved and certain constraints should be imposed on  $\alpha$ , in order for the Einstein frame observables to be compatible with the observational data. For example if  $N = 60$ , the parameter  $\alpha$  should satisfy  $\alpha < 29$ , and for  $N = 50$ , the parameter  $\alpha$  should satisfy  $\alpha < 20$ . Nevertheless we will not further discuss these cases, since it is difficult to obtain analytical solutions in the Jordan frame, for these values of  $\alpha$ .

### C. Limiting Cases of $\alpha$

However, let us briefly discuss the small- $\alpha$  limit of the algebraic equation (27), in order to see how the  $F(R)$  gravity behaves. We shall present only the leading order behavior. Let us start with the leading order result, in which case, the algebraic equation (27) in the limit  $\alpha \ll 1$  becomes,

$$RF_R^{\sqrt{\frac{1}{\alpha}}} - 4\alpha\mu^2 \left( F_R^{\sqrt{\frac{1}{\alpha}}} + \left( \sqrt{\frac{1}{\alpha}} - 2 \right) n \right) = 0, \quad (73)$$

so the solution to this equation is [22],

$$F_R(R) = \frac{R}{4\alpha\mu^2} + n(2 - \frac{1}{\sqrt{2}})R^{1-\frac{1}{\sqrt{\alpha}}}, \quad (74)$$

which is valid for  $0 < \frac{1}{\sqrt{\alpha}}$ . By integrating, we find at leading order that the resulting  $F(R)$  gravity is equal to,

$$F(R) = \frac{R^2}{8\alpha\mu^2} + nR^{2-\frac{1}{\sqrt{\alpha}}}. \quad (75)$$

Since  $R \gg \gamma$ , the leading order that controls the dynamical evolution in the small- $\alpha$  limit is the term  $\sim R^2$ , therefore that is  $R^2$  inflation what drives the evolution. It is interesting that the inflationary  $F(R)$  gravity of Eq. (75) reminds the sector of unified inflation-dark energy  $F(R)$  gravity studied in Refs. [36, 44]. Then, by adding to above approximate expression for alpha-attractor  $F(R)$  inflationary theory the exponential dark energy sector  $F(R) = R - 2\Lambda(1 - e^{R/R_0})$ , we get unification of alpha-attractor inflation with dark energy in  $F(R)$  gravity.

We can easily find the observational indices for the  $F(R)$  gravity of Eq. (75), so by using the first equation in Eq. (31), upon differentiation with respect to the cosmic time and by keeping leading order terms in the slow-roll approximation we get the following differential equation,

$$\frac{9\gamma^2 H(t)H^{(3)}(t)}{\mu^2} + \frac{27\gamma^2 H(t)^2 H''(t)}{\mu^2} + \frac{54\gamma^2 H(t)H'(t)^2}{\mu^2} = 0, \quad (76)$$

and by dividing with  $H(t)^2$  we get,

$$\frac{9\gamma^2 H^{(3)}(t)}{\mu^2 H(t)} + \frac{27\gamma^2 H''(t)}{\mu^2} + \frac{54\gamma^2 H'(t)^2}{\mu^2 H(t)} = 0. \quad (77)$$

So the dominant term in the slow-roll approximation is the second, and by solving the resulting differential equation we obtain the following Hubble rate,

$$H(t) \simeq C_1 - C_2 t, \quad (78)$$

which is valid during the slow-roll era. Hence the resulting evolution is a quasi-de Sitter evolution, and by using the same line of research as in the previous sections, the resulting observational indices are identical to the ones appearing in Eqs. (72) and (52). So actually, the  $R^2$  model is the attractor of all the  $F(R)$  gravity equivalent theories of the Einstein frame  $\alpha$ -attractor models, always in the slow-roll approximation. It is easy to see that due to the relation (20), the Ricci scalar as a function of the canonical scalar field will be  $R = \frac{1}{A}e^{-\sqrt{\frac{2}{3\alpha}}\varphi}$ , so a leading order term in the potential is,

$$V(\varphi) \sim \frac{\sqrt{\alpha}}{A}e^{-\sqrt{\frac{2}{3\alpha}}\varphi}, \quad (79)$$

so indeed, the potential of Eq. (11) is partially reconstructed. As it can be easily checked the leading order term in the potential (79) is generated by the  $R^2$  term in the  $F(R)$  gravity.

Let us discuss another limiting case of  $\alpha$ , in which case  $\alpha$  is too large. In the large- $\alpha$  limit, the algebraic equation (27) becomes approximately,

$$F_R R - 4\alpha F_R^2 \mu^2 (1 - 2n) = 0, \quad (80)$$

so the resulting  $F(R)$  gravity is at leading order,

$$F(R) \simeq \frac{R^2}{8\mu^2(\alpha - 2\alpha n)}. \quad (81)$$

Therefore in this case too, the  $R^2$  model is the attractor of the  $F(R)$  gravities, at least when the slow-roll approximation.

The resulting behavior in all the  $F(R)$  cases we studied indicates that the  $F(R)$  gravity equivalent theories of the Einstein frame  $\alpha$ -attractors models, have a unique attractor when the slow-roll limit is used, and this is the  $R^2$  model. Therefore, regardless how the  $F(R)$  gravity looks at second to leading order, the observational indices are affected mainly by the leading order term in the  $F(R)$  gravity, and this is the reason why the resulting observational indices are identical to the ones of the Starobinsky model.

#### IV. CONCLUSIONS

In this paper we investigated the  $F(R)$  gravity equivalent theory of some classes of Einstein frame  $\alpha$ -attractor models. The full analytic treatment of the problem is not possible, so we chose a convenient Einstein frame  $\alpha$ -attractor model, and we calculated in detail the slow-roll indices in the slow-roll limit of the  $F(R)$  gravity theory. As we demonstrated, in the Jordan frame, the attractors picture remains, since the resulting spectral index of primordial curvature perturbations and the scalar-to-tensor ratio remain attractors of the conveniently chosen  $F(R)$  models. Interestingly enough, the resulting observational indices in the Jordan frame, are identical to the indices of the Starobinsky model, and actually the  $R^2$  model is the attractor in the Jordan frame, at least when the slow-roll approximation is used. This result is not accidental, since in all the cases we studied, the  $F(R)$  gravity in the limit  $R \gg 1$ , are approximately equal to  $\sim R^2$ , so the behavior is similar to the  $R^2$  model.

An important issue is that the scalar-to-tensor ratio in the Jordan frame does not depend on the “ $\alpha$ ” parameter, and this is a difference between the Einstein and Jordan frame description. The question then is, does this occur due to the fact that the slow-roll approximation is used? Do the slow-roll conditions in the two frames impose different conditions on the resulting evolution? The quick answer is no, the two frames are equivalent, but this can be shown explicitly only for the  $R^2$  model. However, for the simple  $\alpha$ -model we used, we showed that this is not true, so the question is if this holds true in general, or it holds true only for the specific models we studied. We defer this task in the future since the lack of analyticity forbids us for the moment to have a definite answer on this.

#### Acknowledgments

This work is supported by MINECO (Spain), project FIS2013-44881 (S.D.O) and by Min. of Education and Science of Russia (S.D.O and V.K.O).

#### Appendix: The Case of an $\alpha$ -dependent Canonical Transformation

Consider the case that the canonical transformation which connects the Einstein and the Jordan frame is  $\alpha$ -dependent. In this case, the transformation (20) becomes,

$$\varphi = \sqrt{\frac{3\alpha}{2}} \ln(F'(A)) \quad (82)$$

In this case, the metric in the Einstein and Jordan frames, namely  $\hat{g}_{\mu\nu}$  and  $g_{\mu\nu}$ , are related as follows,

$$g_{\mu\nu} = e^{-\sqrt{\frac{2}{3\alpha}}\varphi} \hat{g}_{\mu\nu}, \quad (83)$$

where  $g_{\mu\nu}$  denotes the Einstein frame metric. In order to make the presentation more transparent, we will adopt another notation different from the one we used in the main text. Suppose that we identify  $\psi^2 = e^{\sqrt{\frac{2}{3\alpha}}\varphi}$ , then the Ricci scalar transforms as,

$$R = \psi^2 \left( \tilde{R} + 6\tilde{\square}\Psi - 6\tilde{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi \right), \quad (84)$$

where  $\Psi = \ln\psi$ . We can rewrite the Jordan frame  $F(R)$  action (18) as follows,

$$\mathcal{S} = \int d^4x \sqrt{-g} \left( \frac{1}{2} F'(R) R - V \right), \quad (85)$$

where the potential  $V$  is equal to,

$$V = \frac{F'R - F}{2}. \quad (86)$$

The determinant of the metric under the transformation (83) transforms as  $\sqrt{-g} = \psi^{-4}\sqrt{\tilde{g}}$ , where in terms of  $\psi$ , the scalar field is written as  $\varphi = 2\sqrt{\frac{3\alpha}{2}}\ln\psi$ . By combining the above, the resulting Einstein frame action reads,

$$\mathcal{S} = \int d^4x \sqrt{-\tilde{g}} \left( \tilde{R} - 3\tilde{g}^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi - V \right). \quad (87)$$

It can be easily shown that  $\Psi = \varphi/\sqrt{6\alpha}$ , so the Einstein action can be written in terms of the scalar field  $\varphi$ , and we have,

$$\mathcal{S} = \int d^4x \sqrt{-\tilde{g}} \left( \tilde{R} - \frac{1}{2\alpha} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right), \quad (88)$$

where in this case the potential  $V(\varphi)$  is equal to,

$$V(\varphi) = \frac{1}{2} \left( e^{-\sqrt{2/(3\alpha)}\varphi} R \left( e^{\sqrt{2/(3\alpha)}\varphi} \right) - e^{-2\sqrt{2/(3\alpha)}\varphi} F \left[ R \left( e^{\sqrt{2/(3\alpha)}\varphi} \right) \right] \right) \quad (89)$$

Hence by looking the resulting scalar action (88), it can be seen that the  $\alpha$ -dependent conformal transformation leads to a non-canonical scalar-tensor theory. Note that by further re-scaling the scalar field  $\varphi \rightarrow \phi\sqrt{\alpha}$ , one obtains a canonical scalar field theory, however in this case, the canonical transformation (82) becomes,

$$\phi = \sqrt{\frac{3}{2}} \ln(F'(A)), \quad (90)$$

which does not depend on  $\alpha$  and therefore it is identical and it leads to the same results as the transformation we used in Eq. (20).

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